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CAS 701
Fall 2007
Exercise Sheet 1

CAS 701 — Logic and Discrete Mathematics in Software Engineering

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1 Natural Deduction Proofs

For each of the following sequents, produce a formal proof on paper, and transscribe your proof into the language of the “Logic Daemon” proof checker to check your proofs.

Be aware of the distinction between **basic rules** and derived rules, and produce basic-rule-proofs for any derived rules you use.

The following basic rules have not yet been shown in class (“$\bot$”, read also “bottom”, stands for “Falsum”):

$$\frac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{E} \qquad\qquad \frac{\bot}{\phi}\ \bot\mathrm{E} \qquad\qquad \frac{\boxed{\begin{array}{c}\phi\\ \\ \\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{I}$$

In the Logic Daemon, there is no “$\bot$”, and you have to use rule RAA.

(a) $\vdash \ p \to (q \to p)$

(b) $\vdash \ (p \to q) \to ((q \to r) \to (p \to r))$

(c) $p \lor q, \neg q \lor r \ \vdash \ p \lor r$

(d) $p \land \neg p \ \vdash \ \neg(r \to q) \lor (r \to q)$

(e) $p \to q, s \to t \ \vdash \ p \land s \to q \lor t$

(f) $\neg(\neg p \lor q) \ \vdash \ p$

(g) $\vdash \ \neg p \to (p \to (p \to q))$

(h) $\neg(p \to q) \ \vdash \ q \to p$

(i) $\vdash \ ((p \to q) \land (q \to p)) \to ((p \lor q) \to (p \land q))$

(j) $\vdash \ p \lor q \leftrightarrow \neg(\neg p \land \neg q)$

(k) $\vdash \ (p \lor q) \leftrightarrow ((p \to q) \to q)$

Note: Questions 2–4 use semantics that has not yet been presented in class. Please find the standard definitions, for example on the Wikipedia page under the heading "Soundness and completeness of the rules"; where Wikipedia says "S semantically entails ϕ" I use the notation "$S \models \phi$".

2 Invalidity

For each of the following sequents, show that it is not valid by providing a valuation for which all the antecedents evaluate to True, but the consequent evaluates to False.

(a) $\neg m \vee (q \to p) \vdash \neg p \wedge q$

(b) $\neg r \to (p \vee q), r \wedge \neg q \vdash r \to q$

(c) $p \to (q \to r) \vdash p \to (r \to q)$

(d) $\neg p, p \vee q \vdash \neg q$

(e) $p \to (\neg q \vee r), \neg r \vdash \neg q \to \neg p$

(f) $p \to (q \vee r) \vdash (p \to q) \wedge (p \to r)$

3 Semantic Equivalence

Page 54, Def. 1.40: $\phi \equiv \psi$ (read: ϕ is **semantically equivalent** to ψ) iff $\phi \models \psi$ and $\psi \models \phi$.

(a) Show the following semantic equivalences in two principally different ways:

- $\phi \vee (\phi \wedge \eta) \equiv \phi$
- $\phi \wedge (\phi \vee \eta) \equiv \phi$
- $\phi \vee (\psi \wedge \eta) \equiv (\phi \vee \psi) \wedge (\phi \vee \eta)$

(b) Show also the following semantic equivalences:

- $\phi \wedge (\psi \vee \eta) \equiv (\phi \wedge \psi) \vee (\phi \wedge \eta)$
- $\neg(\phi \vee \eta) \equiv \neg\phi \wedge \neg\eta$
- $\neg(\phi \wedge \eta) \equiv \neg\phi \vee \neg\eta$

4 Normal Forms

For each of the following formulae, calculate the indicated normal forms.

(a) CNF(NNF(IMPL_FREE($\neg(p \to (\neg(q \wedge (\neg p \to q))))$)))

(b) CNF(NNF(IMPL_FREE($((p \to \neg q) \to \neg p) \to q$)))

(c) DNF(NNF(IMPL_FREE($(p \to q) \to (q \to p)$)))