The Knuth-Bendix Completion Algorithm

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What's a word problem?

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$\alpha \quad \beta$

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Not easy to solve generally.

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- Operators: $f_1, \ldots, f_N$
- f_k has degree d_k

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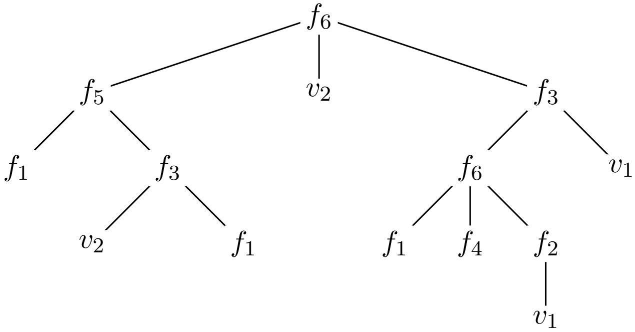
$$
\begin{aligned}
W &\rightarrow v_k \\
W &\rightarrow f_k \underbrace{W \ldots W}_{d_k}
\end{aligned}
$$

Tree structure



$$f_6 f_5 f_1 f_3 v_2 f_1 v_2 f_3 f_6 f_1 f_4 f_2 v_1 v_1$$

Ordering on words

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For an identity $\alpha_k \equiv \beta_k$, assuming $\alpha_k > \beta_k$, we have the *reduction* $\alpha_k \to \beta_k$.

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A set of reductions is *complete* if for any irreducible words $\alpha \neq \beta$, we have $\alpha \not\equiv \beta$.

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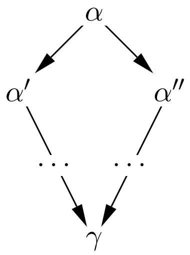
Complete iff the *lattice condition* holds:



Superpositions

$$\sigma(\lambda_1, \mu, \lambda_2)$$

- λ_1 and λ_2 are words
- μ is a subword of λ_2
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- Replace the μ in λ_2 with λ_1 to get $\sigma(\lambda_1, \mu, \lambda_2)$
- $\sigma(\lambda_1, \mu, \lambda_2)$ must "look like" λ_2

Let's try it. . .

$$e \cdot a \rightarrow a \quad (1)$$

$$a^{-} \cdot a \rightarrow e \quad (2)$$

$$(a \cdot b) \cdot c \rightarrow a \cdot (b \cdot c) \quad (3)$$

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$$a^{-} \cdot (a \cdot b) \rightarrow b \tag{4}$$

Let's try it...

$$
\begin{aligned}
e \cdot a &\rightarrow a && (1)\\
a^{-} \cdot a &\rightarrow e && (2)\\
(a \cdot b) \cdot c &\rightarrow a \cdot (b \cdot c) && (3)\\
a^{-} \cdot (a \cdot b) &\rightarrow b && (4)\\
e^{-} \cdot a &\rightarrow a && (5)
\end{aligned}
$$

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$$a^{-} \cdot (a \cdot b) \rightarrow b \quad (4)$$

$$e^{-} \cdot a \rightarrow a \quad (5)$$

etc.

Until finally. . .

$$
\begin{array}{llcl llcl}
(1) & e \cdot a & \rightarrow & a & (9) & e^{-} & \rightarrow & e \\
(2) & a^{-} \cdot a & \rightarrow & e & (10) & a^{--} & \rightarrow & a \\
(3) & (a \cdot b) \cdot c & \rightarrow & a \cdot (b \cdot c) & (11) & a \cdot a^{-} & \rightarrow & e \\
(4) & a^{-} \cdot (a \cdot b) & \rightarrow & b & (13) & a \cdot (a^{-} \cdot b) & \rightarrow & b \\
(8) & a \cdot e & \rightarrow & a & (20) & (a \cdot b)^{-} & \rightarrow & b^{-} \cdot a^{-}
\end{array}
$$